Discrete Mathematics and Applications

COT3100

Dr. Ungor

Sources: Slides are based on Dr. G. Bebis' material.

Foundations of Logic: Overview

- Propositional logic: (Sections 1.1-1.3)
  - Basic definitions.
  - Equivalence rules & derivations.
- Predicate logic (Section 1.4)
  - Predicates.
  - Quantified predicate expressions.
  - Equivalences & derivations.

Propositional Logic

Propositional Logic is the logic of compound statements built from simpler statements using *Boolean connectives.*

Applications:

- Design of digital electronic circuits.
- Expressing conditions in computer programs.
- Queries to databases & search engines.

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- What are **Propositions**?
- What are **Boolean connectors**?

Definition of a *Proposition*

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- A *proposition* (*p*, *q*, *r*, …) is simply a *statement* (*i.e.*, a declarative sentence) *with a definite meaning*, having a *truth value* that's either *true* (T) or *false* (F) (**never** both, neither, or somewhere in between).

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- In *probability theory,* we assign *degrees of certainty* to propositions. For now: True/False only!

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- "Yeah, I sorta dunno, whatever..." (vague)

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- "La la la la la." (meaningless interjection)
- "Just do it!" (imperative, command)
- "Yeah, I sorta dunno, whatever..." (vague)
- "1 + 2" (expression with a non-true/false value)

Boolean operators

- Negation (NOT)
- Conjunction (AND)
- Disjunction (OR)
- Exclusive-Or (XOR)
- Implication (IF)
- Bi-conditional (IFF)

Operators / Connectives

- An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (*e.g.*, "+" in numeric exprs.)
- *Unary* operators take 1 operand (*e.g.,* -3); *binary* operators take 2 operands (*e.g.,* 3 × 4).
- *Propositional* or *Boolean* operators operate on propositions or truth values instead of on numbers.

The Negation Operator

- The unary *negation operator* "¬" (*NOT*) transforms a prop. into its logical *negation*.
- Example: If p = "I have brown hair.", then $\neg p$ = "I do **not** have brown hair."
- *Truth table* for NOT:

p	$\neg p$
T	F
F	T

The Conjunction Operator

- The binary *conjunction operator* "$\wedge$" (*AND*) combines two propositions to form their logical *conjunction*.

- *E.g.,* If p="I will have salad for lunch." and q="I will have steak for dinner.", then $p \wedge q$="I will have salad for lunch **and** I will have steak for dinner."

Conjunction Truth Table

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Conjunction Truth Table

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

- ¬ and ∧ operations together are universal, i.e., sufficient to express *any* truth table!

The Disjunction Operator

- The binary *disjunction operator* "∨" (*OR*) combines two propositions to form their logical *disjunction*.

p="That car has a bad engine."

q="That car has a bad carburetor."

$p \vee q$="Either that car has a bad engine, **or** that car has a bad carburetor."

Disjunction Truth Table

p	q	$p \vee q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Disjunction Truth Table

- Note that $p \vee q$ means that p is true, or q is true, **or both** are true!
- So this operation is also called *inclusive or,* because it **includes** the possibility that both p and q are true.
- "¬" and "∨" together are also universal.

p	q	$p \vee q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

A Simple Exercise

Let p="It rained last night",
q="The sprinklers came on last night,"
r="The lawn was wet this morning."

Translate each of the following into English:

$\neg p$ =

$r \land \neg p$ =

$\neg\, r \lor p \lor q$ =

A Simple Exercise

Let p=“It rained last night”,
q=“The sprinklers came on last night,”
r=“The lawn was wet this morning.”

Translate each of the following into English:

$\neg p$ = “It didn’t rain last night.”

$r \wedge \neg p$ =

$\neg r \vee p \vee q$ =

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Translate each of the following into English:

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$r \wedge \neg p$ = "The lawn was wet this morning, and it didn't rain last night."

$\neg r \vee p \vee q$ =

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$\neg r \vee p \vee q$ =

"Either the lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

The *Exclusive Or* Operator

- The binary *exclusive-or operator* "$\oplus$" (*XOR*) combines two propositions to form their logical "exclusive or" (exjunction?).

p = "I will earn an A in this course,"

q = "I will drop this course,"

$p \oplus q$ = "I will either earn an A for this course, or I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called *exclusive or,* because it **excludes** the possibility that both p and q are true.
- "¬" and "⊕" together are **not** universal.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Natural Language is Ambiguous

Note that in English "or" is *by itself* ambiguous regarding the "both" case!

"Pat is a singer or Pat is a writer." -

"Pat is rich or Pat is poor." -

p	q	p or q
F	F	F
F	T	T
T	F	T
T	T	undef.

Need context to disambiguate the meaning!

For this class, assume "or" means inclusive.

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The *Implication* Operator

- The *implication* $p \rightarrow q$ states that p implies q.
- It is FALSE only in the case that p is TRUE but q is FALSE.

e.g., p="I am elected."
q="Taxes will be lowered."

$p \rightarrow q$ = "If I am elected, then taxes will be lowered" (else it could go either way)

The *Implication* Operator

- Terminology for the structure of implication $p \rightarrow q$
- p: *Hypothesis (antecedent or premise)*
- q: *Conclusion (consequence)*

Implication Truth Table

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** imply that p causes q!

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F	F	T
F	T	T
T	F	F
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Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** imply that p causes q!
- $p \rightarrow q$ does **not** imply that p or q **are ever true**!

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F	T	T
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- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** imply that p causes q!
- $p \rightarrow q$ does **not** imply that p or q **are ever true**!
- *E.g.* "(1=0) → pigs can fly" is TRUE!

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Examples of Implications

- “If this lecture ends, then the sun will rise tomorrow.” *True* or *False*?
- “If Monday is a day of the week, then I am a not teacher.” *True* or *False*?
- “If 1+1=6, then George passed the exam.” *True* or *False*?
- “If the moon is made of green cheese, then I am richer than Bill Gates.” *True* or *False*?

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Inverse, Converse, Contrapositive

Some terminology:

- The *inverse* of $p \rightarrow q$ is: $\neg p \rightarrow \neg q$
- The *converse* of $p \rightarrow q$ is: $q \rightarrow p$.
- The *contrapositive* of $p \rightarrow q$ is: $\neg q \rightarrow \neg p$.

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Contrapositive

How do we know for sure?

Proving the equivalence of $p \rightarrow q$ and its contrapositive using truth tables:

p	q	$\neg q$	$\neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

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- p = "It is raining."

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- It is TRUE only when both $p \rightarrow q$ and $q \rightarrow p$ are TRUE.
- p = "It is raining."
- q = "The home team wins."

The *biconditional* operator

The *biconditional* $p \leftrightarrow q$ states that p is true *if and only if (IFF)* q is true.

- It is TRUE only when both $p \rightarrow q$ and $q \rightarrow p$ are TRUE.
- p = "It is raining."
- q = "The home team wins."
- $p \leftrightarrow q$ = "If and only if it is raining, the home team wins."

Biconditional Truth Table

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Biconditional Truth Table

- $p \leftrightarrow q$ means that p and q have the **same** truth value.

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  - Note this truth table is the exact **opposite** of $\oplus$'s!

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  $p \leftrightarrow q$ means $\neg(p \oplus q)$

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Biconditional Truth Table

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    - Note this truth table is the exact **opposite** of $\oplus$'s!

      $p \leftrightarrow q$ means $\neg(p \oplus q)$
- $p \leftrightarrow q$ does **not** imply p and q are true, or cause each other.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

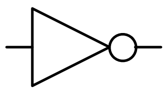
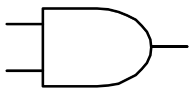
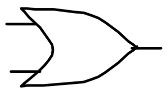
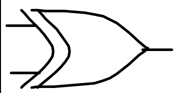
Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Nested Propositional Expressions

- Use parentheses to *group sub-expressions*:
  "I just saw my old ***f***riend, and either he's ***g***rown or I've ***s***hrunk." = $f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
    - $f \wedge g \vee s$ would be ambiguous
- By convention, "¬" takes *precedence* over both "∧" and "∨".
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, **not** $\neg (s \wedge f)$

Some Alternative Notations

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\rightarrow$	$\leftrightarrow$
Boolean algebra:	$\overline{p}$	pq	$+$	$\oplus$		
C/C++/Java (wordwise):	!	&&	||	!=		==
C/C++/Java (bitwise):	~	&	|	^		
Logic gates:						

Bits and Bit Operations

- A *bit* is a binary (base 2) digit: 0 or 1.
- Bits may be used to represent truth values.
- By convention: 1 represents "true";
  0 represents "false".

- *Boolean algebra* is like ordinary algebra except that variables stand for bits, + means "or", and multiplication means "and".

Bit Strings

- A *Bit string* of *length n* is an ordered series or sequence of $n \geq 0$ bits.
- By convention, bit strings are written left to right: *e.g.* the first bit of "1001101010" is 1.
- When a bit string represents a base-2 number, by convention the first bit is the *most significant* bit. *Ex.* $1101_2 = 8+4+1 = 13$.

Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.
- E.g.:
  01 1011 0110
  11 0001 1101
  11 1011 1111 Bit-wise OR
  01 0001 0100 Bit-wise AND
  10 1010 1011 Bit-wise XOR

Propositional Equivalences

Propositional Equivalence

Two *syntactically* (*i.e.,* textually) different compound propositions may be *semantically* identical (*i.e.,* have the same meaning). We call them *equivalent*.

NEXT:

- Learn about various *equivalence rules* or *laws*.
- Learn how to *prove* equivalences using *symbolic derivations*.

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Ex. $p \lor \neg p$ [What is its truth table?]

A *contradiction* is a comp. prop. that is **false** no matter what! *Ex.* $p \land \neg p$ [Truth table?]

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Ex. $p \vee \neg p$ [What is its truth table?]

A *contradiction* is a comp. prop. that is **false** no matter what! *Ex.* $p \wedge \neg p$ [Truth table?]

Other comp. props. are *contingencies*.

Proving Equivalences

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Compound proposition P is *logically equivalent* to compound proposition Q, written $P \Leftrightarrow Q$, **IFF** the compound proposition $P \leftrightarrow Q$ is a tautology.

Proving Equivalence via Truth Tables

P = $p \vee q$, Q = $\neg(\neg p \wedge \neg q)$

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F					
F	T					
T	F					
T	T					

Proving Equivalence via Truth Tables

$$\overbrace{p \vee q}^{P} \Leftrightarrow \overbrace{\neg(\neg p \wedge \neg q)}^{Q}$$

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

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F	T	T				
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Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
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Equivalence Laws

- These are similar to the <u>arithmetic identities</u> you may have learned in algebra, but for propositional equivalences instead.
- They provide a <u>pattern or template</u> that can be used to match much more complicated propositions and to find equivalences for them.

Equivalence Laws - Examples

- *Identity*: $p \wedge \mathbf{T} \Leftrightarrow p$ $p \vee \mathbf{F} \Leftrightarrow p$
- *Domination*: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$ $p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- *Idempotent*: $p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$
- *Double negation:* $\neg\neg p \Leftrightarrow p$
- *Commutative:* $p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$
- *Associative:* $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

- *De Morgan's*:

  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

More Equivalence Laws

- *Absorption*:

  $p \vee (p \wedge q) \Leftrightarrow p$

  $p \wedge (p \vee q) \Leftrightarrow p$

- *Trivial tautology/contradiction*:

  $p \vee \neg p \Leftrightarrow \mathbf{T}$ $\quad$ $p \wedge \neg p \Leftrightarrow \mathbf{F}$

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Implication: $p \rightarrow q \Leftrightarrow \neg p \vee q$
- Biconditional: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

  $p \leftrightarrow q \Leftrightarrow \neg (p \oplus q)$
- Exclusive or: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg (p \wedge q)$

  $p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

An Example Problem

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- Check using a symbolic derivation whether $(p \wedge \neg q) \rightarrow (p \oplus r) \Leftrightarrow \neg p \vee q \vee \neg r$.

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$(p \wedge \neg q) \underline{\rightarrow} (p \oplus r) \Leftrightarrow$

[Expand definition of →] $\neg(p \wedge \neg q) \vee (p \underline{\oplus} r)$

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$(p \wedge \neg q) \underline{\rightarrow} (p \oplus r) \Leftrightarrow$

[Expand definition of →] $\neg(p \wedge \neg q) \vee (p \underline{\oplus} r)$

[Defn. of ⊕] $\Leftrightarrow \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

An Example Problem

- Check using a symbolic derivation whether $(p \land \neg q) \to (p \oplus r) \Leftrightarrow \neg p \lor q \lor \neg r$.

$(p \land \neg q) \underline{\to} (p \oplus r) \Leftrightarrow$

[Expand definition of →] $\neg(p \land \neg q) \lor (p \underline{\oplus} r)$

[Defn. of ⊕] $\Leftrightarrow \neg(p \land \neg q) \lor ((p \lor r) \land \neg(p \land r))$

[DeMorgan's Law]

An Example Problem

- Check using a symbolic derivation whether $(p \wedge \neg q) \rightarrow (p \oplus r) \Leftrightarrow \neg p \vee q \vee \neg r$.

$(p \wedge \neg q) \underline{\rightarrow} (p \oplus r) \Leftrightarrow$

[Expand definition of →] $\neg(p \wedge \neg q) \vee (p \underline{\oplus} r)$

[Defn. of ⊕] $\Leftrightarrow \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

[DeMorgan's Law]

$\Leftrightarrow (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

An Example Problem

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$(p \wedge \neg q) \underline{\to} (p \oplus r) \Leftrightarrow$

[Expand definition of $\to$] $\neg(p \wedge \neg q) \vee (p \underline{\oplus} r)$

[Defn. of $\oplus$] $\Leftrightarrow \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

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$\Leftrightarrow (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

Example Continued...

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$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \Leftrightarrow$ [∨ commutes]

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$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \Leftrightarrow$ [∨ commutes]

$\Leftrightarrow \underline{(q \vee \neg p)} \vee ((p \vee r) \wedge \neg(p \wedge r))$ [∨ associative]

Example Continued...

$(\neg p \lor q) \lor ((p \lor r) \land \neg(p \land r)) \Leftrightarrow$ [∨ commutes]

$\Leftrightarrow \underline{(q \lor \neg p)} \lor ((p \lor r) \land \neg(p \land r))$ [∨ associative]

$\Leftrightarrow q \lor \underline{(\neg p \lor} ((p \lor r) \land \neg(p \land r)))$ [distrib. ∨ over ∧]

Example Continued...

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$\Leftrightarrow q \lor (\underline{\neg p \lor} ((p \lor r) \land \neg(p \land r)))$ [distrib. ∨ over ∧]

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Example Continued...

$(\neg p \lor q) \lor ((p \lor r) \land \neg(p \land r)) \Leftrightarrow$ [∨ commutes]

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[trivial taut.] $\Leftrightarrow q \lor ((\underline{\mathbf{T}} \lor r) \land (\neg p \lor \neg(p \land r)))$

Example Continued...

$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \Leftrightarrow$ [∨ commutes]

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[domination] $\Leftrightarrow q \vee (\underline{\mathbf{T}} \wedge (\neg p \vee \neg(p \wedge r)))$

[identity] $\Leftrightarrow q \vee (\neg p \vee \neg(p \wedge r)) \Leftrightarrow$ *cont.*

End of Long Example

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$q \vee (\neg p \vee \neg(p \wedge r))$

End of Long Example

$q \lor (\neg p \lor \neg(p \land r))$

[DeMorgan's] $\Leftrightarrow q \lor (\neg p \lor (\neg p \lor \neg r))$

End of Long Example

$q \vee (\neg p \vee \neg(p \wedge r))$

[DeMorgan's] $\Leftrightarrow q \vee (\neg p \vee (\neg p \vee \neg r))$

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Q.E.D. (quod erat demonstrandum)