PCPs and Inapproximability CIS 6930	Sep 17^{th}, 2009
Hardness of Vertex Cover and Steiner tree problem	
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1 Hardness of Vertex Cover problem

In this section, we denote

- **VC(d)** the restriction of the cardinality Vertex Cover problem to instances in which each vertex has degree *at most d*.
- **MAX-3SAT(d)** the restriction of MAX-3SAT to Booldean formulae in which each variable occurs *at most d times*.

Theorem 29.13

There is a gap-preserving reduction from MAX-3SAT(29) to VC(30) that transforms a Boolean formula ϕ to a graph G = (V,E) such that

- *If $OPT(\phi) = m$, then $OPT(G) \leq \frac{2}{3}|V|$, and*
- *If $OPT(\phi) < (1 - \epsilon_b)m$, then $OPT(G) > (1 + \epsilon_v)\frac{2}{3}|V|$ where $\epsilon_v = \frac{\epsilon_b}{2}$*

Proof:
In order to prove this Theorem, we need to construct a transformation between instances of MAX-3SAT(29) and VC(30), or, in other words, we need to transform a Boolean formula ϕ of MAX-3SAT(29) to a graph G = (V,E) of VC(30) in such a way that the two conditions above will follow. Without loss of generality, we can assume that ϕ has m clauses of exactly 3 literals in CNF form each. Below is the construction of an instance G=(V,E) from ϕ

1. Let each literal in ϕ be a vertex in V.
2. For each clause, G has 3 edges connecting its 3 vertices, and
   For any u, v $\in$ V, if u and v are negations of each other, there will be an edge connecting u and v.

For example, the Boolean formula $\phi = (x_1 \vee \bar{x_2} \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee x_3)$ has the corresponding graph G as depicted in Figure 1. Besides, using this standard construction, we observe that

- ϕ has m clauses of exactly 3 literals each and a literal has a corresponding verex in V. Thus V has exactly 3m vertices ($|V| = 3m$).

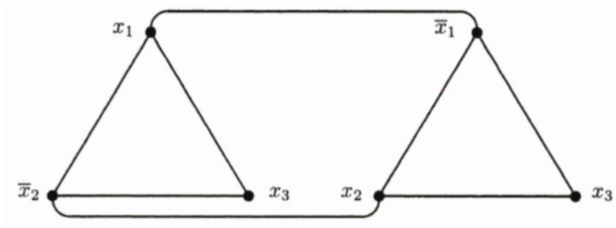

Figure 1: The graph constructed by $\phi = (x_1 \vee \bar{x_2} \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee x_3)$

- Each vertex in G has 2 edges of type #1 and at most 28 edges of type #2. Therefore, the vertex degree is at most 30.
- Any Maximum Independent Set (MIS) in G has size at most m (since it can pick up at most a vertex in each "triangle").

Showing directly this construction carries the two inequalitis is difficult, we should show that via the MIS (since the minimum vertex cover is the complement of the MIS). In particular, we'll show that the size of a MIS is percisely OPT(ϕ). ($\geq$) Condsider the optimal assignment for ϕ, i.e. the assignment satisfies OPT(ϕ) clauses. We then pick a satisfied literal from each clause and consider the set of corresponding verices in G. That set is an independet set of G (since every vertex is satisfied, there would not be an edge of type #2 connecting any pair of them) and its size is not bigger than the MIS, which implies $OPT(\phi) \leq |I|$. ($\leq$) Conversely, let I be an independet set of G, we show that $|I| \leq OPT(\phi)$. Setting all literals corresponding to I true will give an assignment that satisfies at least $|I|$ clauses, and of course, this number should not exceed the maximum satified clauses OPT(ϕ): $|I| \leq OPT(\phi)$. Now,

- If OPT(ϕ) = m, i.e, a MIS of G has size of m, then $OPT(G) = 2m \leq \frac{2}{3}|V|$.
- If $OPT(\phi) < (1-\epsilon_b)m$, then $OPT(G) > 3m - (1-\epsilon_b)m = (1+\frac{\epsilon_b}{2})\frac{2}{3}|V|$.

Hence, we can not hope to have an approximation algorithm for VC(30) with an approximation guarantee of $(1+\frac{\epsilon_b}{2})$, assuming $P \neq NP$
■

Question: Do we really need the two numbers 29 and 30 in this problem? Will the proof still valid without these numbers?

2 Hardness of Steiner Tree problem

Theorem 29.14

There is a gap preserving reduction from VC(30) to the Steiner tree problem. It transforms an instance G(V,E) of VC(30) to an instance H = (R,S,cost) of Steiner tree, where R and S are required and Steiner vertices of H, and cost is a metric in R. It satisfies:

- *If $OPT(G) \leq \frac{2}{3}|V|$, then $OPT(H) \leq |R| + \frac{2}{3}|S| - 1$, and*
- *If $OPT(G) > (1+\epsilon_v)\frac{2}{3}|V|$, then $OPT(H) > (1+\epsilon_s)\big(|R| + \frac{2}{3}|S| - 1\big)$*

where $\epsilon_s = \frac{4\epsilon_v}{97}$

Proof:

We need to construct a transformation from a graph G=(V,E) of VC(30) to a graph H=(R,S,cost) in such a way that G has a vertex cover of size c if and only if H has a Steiner tree of cost $|R|+c-1$. If such transformation exists, we would be able to prove the two proposed inequalities.

Contructing the transformation

Input: A graph G=(V,E)

Output: A graph H=(R,S,cost) where cost is a weight function on edges.

1. For any edge e in E, there is a corresponding vertex r_e in R. ($|R| = |E|$)
2. For any vertex v in V, there is a corresponding vertex s_v in S. ($|S| = |V|$)
3. For any r_e, $r_{e'}$ $\in$ R, there is an edge connects them with cost$(r_e, r_{e'}) = 2$.
4. For any s_v, $s_{v'}$ $\in$ S, there is an edge connects them with cost$(s_v, s_{v'}) = 1$.
5. For any $r_e \in R$ and $s_v \in S$, there is an edge connects them with cost cost$(r_e, s_v) = 1$ if e is incident to v in G.
6. For any $r_e \in R$ and $s_v \in S$, there is an edge connects them with cost cost$(r_e, s_v) = 2$ if e is not incident to v in G.

Claim: Using the above transformation, G has a Vertex Cover of size c iff H has a Steiner tree of size $|R| + c - 1$

($\Rightarrow$) Suppose that G has a Vertex Cover VC of size c. Creat a graph G' whose vertice are in $\{s_v|v \in VC\} \cup \{r_e|e \in E\}$ and whose edges are those of cost 1 in G. Clearly, G' has $|E| + c = |R| + c$ nodes. Moreover, G' is connected since any s_v and $s_{v'}$ are connected (#4) and any r_e is connected to some s_v (for VC is a vertex cover in G, any edge in E must incident to at least a vertex in VC). Therefore, any spanning tree of G' will be a Steiner tree for H (since it covers $R \cup VC$) and has cost of $|R| + c - 1$.

($\Leftarrow$) Assume that H has a Steiner tree T of cost $|R| + c - 1$, we show that G has a Vertex Cover VC of size c. What we hope to have is a Steiner tree with all unit cost edges; however, T may contain edges with cost 2 at the beginning. Therefore, we need to modify T so that it has no edge of cost 2. The following procedure does the job: (For notation convention, we denote [v] and [u,v] the nodes in H corresponding to v $\in$ V and (u,v) $\in$ E in G)

1. If T contains an edge of cost 2 connecting [w] and [u,v], we remove this edge and add two new edges ([w], [u]) and ([u], [u,v]).
2. If T contains an edge of cost 2 connecting [u,v] and [v,w], we remove this edge and add two new edges ([u,v], [v]) and ([v], [v,w]).
3. If T contains and edge of cost 2 connecting [u,v] and [w,z] then removing this arc in T will result in partitioning E into two subsets. Since G is connected, there must be two edges on different sides of the partition that share an enpoint. Let they be (x,y) and (y,t). We then connect ([x,y], [y]) and ([y], [y,t]).

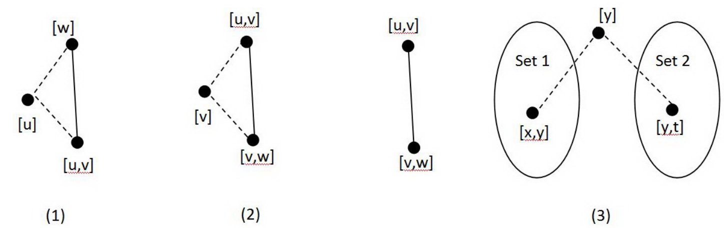

Figure 2: Illustration of modifyding Steiner tree T case (1), (2) and (3)

An illustration for the above procedure is presented in Figure 2. Since the proposed procedure neither disconnects T nor increases its cost, T is still a Steiner tree of H with cost $|R| + c - 1$ containing only unit edges. Clearly, T must span $|R| + c$ nodes including the required set R, and thus, T covers c nodes in S. We show that this set VC of c vertices is a Vertex Cover in G. This is true because any vertex [u,v] is connected to the T by an edge of cost 1, which means that either [u] or [v] is in the tree, i.e, either u or v is in VC. Therefore, VC is a Vertex Cover in G and

- If $OPT(G) \leq \frac{2}{3}|V|$, then $OPT(H) \leq |R| + \frac{2}{3}|S| - 1$
- If $OPT(G) > (1 + \epsilon_v)\frac{2}{3}|V|$, then $OPT(H) > |R| + (1 + \epsilon_v)\frac{2}{3}|S| - 1$

Hence, we can not hope to have an approximation algorithm for Steiner tree problem with an approximation ratio of $(1 + \epsilon)$, for a constant ϵ, assuming $P \neq NP$ ■

Question: Again, can we ignore the number 30 and just simply use VERTEX-COVER for the transformation? How can we prove the last inequality, that is $OPT(H) > (1 + \epsilon_s)\big(|R| + \frac{2}{3}|S| - 1\big)$

References

[1] V.V. Vazirani, Approximation Algorithms, Springer, 2001.

[2] Professor Lucas Trevisan, Lecture notes 3 and 4 in CS294, UC Berkeley, 2006.