CAP5416: Midterm Exam I

23rd October 2001

1. [25 points] **Canny edge detection:** The first derivative of the Gaussian is just one of several choices available for the linear filter/operator that is convolved with the image. Assume that the Gaussian is replaced by $g(x, y) = \frac{1}{1+(x^2+y^2)/\sigma^2}$. The filter in the x-direction is the first derivative $g_x(x, y) = \frac{\partial g}{\partial x}$ and the filter in the y-direction is the first derivative $g_y(x, y) = \frac{\partial g}{\partial y}$.

   (a) [5 points] Determine $g_x(x, y)$ and $g_y(x, y)$. Do the shapes of these two filter/operators look like what you would expect for step edge detection.

   (b) [5 points] We would like to determine the width of the filter W by enforcing the following criterion on $g(x; \sigma) = \frac{1}{1+x^2/\sigma^2}$. (Note that we have switched to one dimensional notation here and also, please note that $g(x; \sigma)$ is not to be confused with its derivative which is actually used in the convolution. Finally, please note that the filter ranges from $-W$ to W and we can use symmetry arguments to solely focus on the positive x-axis.) The criterion is: For $x > W$, $g(x; \sigma) < T$. Determine the minimum value of W as a general function of σ and T and evaluate this minimum value for $\sigma = 1$ and $\sigma = 0.1$. In all calculations, assume $T = 0.01$.

   (c) [5 points] Repeat (b) for the Gaussian $g(x; \sigma) = \exp(-\frac{x^2}{\sigma^2})$. Note that we have simplified the form of the Gaussian from the one that you used in the experiments. For $\sigma = 1$ and $\sigma = 0.1$, and for $T = 0.01$, do you get larger or smaller values for the minimum value of W. Discuss the differences between the two $g(x; \sigma)$ in this context of minimum filter width W.

   (d) [5 points] Irrespective of the form of the filter used, I hope that you understood the importance of hysteresis and non-maximum suppression by actually performing the experiments. Assume that you are implementing hysteresis by rescuing an edge *only* if both the neighboring pixels along the edge have gradients greater than $threshold_{high}$ and the current pixel under consideration has a gradient greater than $threshold_{low}$. Now, we would like to examine what happens if you iterate the hysteresis step. That is, after one pass of hysteresis over the image, you repeat the hysteresis step. Does anything change, i.e. do additional pixels get rescued from the previous iteration?

(e) [5 points] Now assume that you are implementing hysteresis by rescuing an edge if *either* of the neighboring pixels along the edge have gradients greater than $threshold_{high}$ and the current pixel under consideration has a gradient greater than $threshold_{low}$. Now, we would like to examine what happens if you iterate the hysteresis step. That is, after one pass of hysteresis over the image, you repeat the hysteresis step. Does anything change from the previous iteration in this case? Discuss.

2. [30 points] **Hough Transform:** We would like to use the Hough transform for template matching. Assume that we have a template consisting of N points in 2D. For instance, these N points could correspond to the outline of a shape. Assume that the data consists as usual of a set of points $M \gg N$. We wish to set up a voting scheme in the following manner. The template will be rotated and translated across the image plane but it will not be scaled. In this way, we fix the number of parameters in the search space to 3 (one rotation and two translation parameters). For each configuration of the template corresponding to a unique choice of rotation and translation parameters, we have to determine the number of image edge points that vote for it.

(a) [3 points] Suppose, we count a single vote if a template point happens to lie exactly on an image edge point. Then, the total number of votes obtained for a given configuration is exactly equal to the number of template points that happen to exactly lie on top of image edge points. Explain when we get zero votes for a particular configuration of the template.

(b) [7 points] Let us now relax the notion of a vote. Let each image edge point find its nearest neighbor in the template for a given configuration (choice of rotation and translation parameters). Then, the image point $i \in \{1, \ldots M\}$ casts a weighted vote for its chosen template point with the weight given by $w_i = \frac{1}{1+r_i}$ where r_i is the distance from the image edge point i to its nearest neighbor template point. Evaluate the total number of operations in an order of magnitude sense (nearest neighbor calculations + vote accumulation) to compute the votes for a single configuration in terms of M and N.

(c) [5 points] Write down an expression for the Euclidean distance r_{ij} between any image point $u_i, i \in 1, \ldots M$ and any template point $v_j, j \in 1, \ldots N$ for a given configuration [rotation angle θ and translation (t_x, t_y)]. You may assume that each image point and each template point has an $x-$ and $y-$ coordinate.

(d) [5 points] Write down an expression for the nearest neighbor corresponding to that template point $v_{j^*(i)}$ which is closest to an image point u_i. The notation $j^*(i)$ denotes that the nearest neighbor will be dependent on the image point index i.

(e) [5 points] Assuming a nearest neighbor $j^*(i)$, write down the total weighted vote for a particular configuration (fixed choice of rotation and translation). The weighted vote will be in terms of $w_{ij^*(i)} = \frac{1}{1+r_{ij^*(i)}}$. This question is really independent of (c) and (d) above.

(f) [5 points] Are clashes possible, i.e. can different image edge points end up voting for the same template point for a given configuration (fixed rotation and translation parameters). Discuss.

3. [10 points] **Snakes and deformable contours:** It is well known that the snake energy function can unnecessarily penalize moderate to large curvatures when present in the image. If the snake energy contains the term $\sum_i |p_{i+1} - 2p_i + p_{i-1}|^2$ the culprit is the square on the discretized second derivative. How would you replace this term by an equivalent term that does not quite penalize large curvatures to the same degree as the above energy term? [Hint: Consider replacing the square of the discretized second derivative by just the absolute value or some such similar function of the discretized second derivative.] Explain your answer.

4. [25 points] **Calculus of variations:**

(a) [15 points] For each of the following functionals, please write down the corresponding Euler-Lagrange equation.

i. [4 points] $E(f) = \int_0^1 f(x)f''(x)dx$

ii. [4 points] $E(f) = \int_0^1 [f(x) - d(x)]^2 + \lambda \int_0^1 [f''(x)]^2 dx$. Note that $d(x)$ is given and not to be estimated.

iii. [7 points] $E(f) = \int_0^1 \log\left(1 + [f'(x)]^2\right) dx$

(b) [10 points] If $E(f) = 0.5 \int_0^1 \left[f^2(x) + (f'(x))^2\right] dx$, write down the corresponding Euler-Lagrange equation. Discretize the Euler-Lagrange equation. Extra credit: [10 points] Can you show that a discretization of the functional $E(f)$ leads to the same difference equation as the above discretization of the Euler-Lagrange equation.