Homework #2 Solutions

1. **Canny edge detector:** Show that the product of the Canny detection and localization criteria

$$\mathrm{SNR}(f) \times \mathrm{Loc}(f) = \frac{|\int_{-W}^{W} G(-x)f(x)dx|| \int_{-W}^{W} G'(-x)f'(-x)dx|}{n_0\sqrt{\int_{-W}^{W} f^2(x)dx} n_0\sqrt{\int_{-W}^{W} f'^{(2)}(x)dx}}$$

is maximized by $f(x) = G(-x)$.

Use the Cauchy-Schwarz inequality $(\int f(x)g(x)dx)^2 \leq \int f^2(x)dx \int g^2(x)dx$ with equality at $f(x) = cg(x)$ and write

$$\mathrm{SNR}(f) \times \mathrm{Loc}(f) = \frac{|\int_{-W}^{W} G(-x)f(x)dx|| \int_{-W}^{W} G'(-x)f'(-x)dx|}{n_0\sqrt{\int_{-W}^{W} f^2(x)dx} n_0\sqrt{\int_{-W}^{W} f'^{(2)}(x)dx}} \leq \frac{1}{n_0^2}|\int_{-W}^{W} G^2(-x)dx|| \int_{-W}^{W} G'^2(-x)dx|$$

with equality at $f(x) = G(-x)$.

2. **Canny edge detector:** Rewrite the detection and localization criteria for a filter $f_w(x) = f(x/w)$. Show that the product of the detection and localization criteria is *invariant* to w.

$$\begin{aligned}
\mathrm{SNR}(f) \times \mathrm{Loc}(f) &= \frac{|\int_{-W}^{W} G(-x)f(x/w)dx|| \int_{-W/w}^{W/w} G'(-x)f'(-x/w)dx|}{n_0\sqrt{\int_{-W}^{W} f^2(x/w)dx} n_0\sqrt{\int_{-W}^{W} f'^{(2)}(x/w)dx}} \\
&= \frac{w|\int_{-W/w}^{W/w} G(-yw)f(y)dy|| \int_{-W/w}^{W/w} G'(-yw)f'(-y)dy|}{n_0\sqrt{w\int_{-W/w}^{W/w} f^2(y)dy} n_0\sqrt{\frac{1}{w}\int_{-W/w}^{W/w} f'^{(2)}(y)dy}} \\
&= \frac{|\int_{-W/w}^{W/w} G(-yw)f(y)dy|| \int_{-W/w}^{W/w} G'(-yw)f'(-y)dy|}{n_0\sqrt{\int_{-W/w}^{W/w} f^2(y)dy} n_0\sqrt{\int_{-W/w}^{W/w} f'^{(2)}(y)dy}}.
\end{aligned}$$

So, the product is actually not exactly invariant to w. However, it is approximately invariant.

3. **Level sets:** The differential equation obeyed by $x(t)$ is

$$\frac{dx}{dt} = -\frac{x+t}{t+1}.$$

Assuming that $\psi(x,t) = ax^2 + bxt + ct^2 + dx + et + f$ we get

$$\frac{\partial \psi}{\partial x} = 2ax + bt + d$$

and

$$\frac{\partial \psi}{\partial t} = bx + 2ct + e.$$

If an embedding can be found such that

$$\begin{aligned}\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}\frac{dx}{dt} &= 0\\ &\Rightarrow \quad \frac{dx}{dt}=-\frac{\frac{\partial\psi}{\partial t}}{\frac{\partial\psi}{\partial x}}=-\frac{bx+2ct+e}{2ax+bt+d}=-\frac{x+t}{t+1}.\end{aligned}$$

From this, we can identify $a=0$, $b=1$, $c=0.5$, $d=1$, and $e=0$ giving

$$\psi(x,t)=xt+0.5t^2+x+f.$$

Since we want level sets of $\psi(x,t)$, these correspond to

$$\begin{aligned}xt+0.5t^2+x+f &= c\\ &\Rightarrow \quad x(t)=\frac{(c-f)-0.5t^2}{t+1}\end{aligned}$$

If we choose $f=0$ and we seek the zero level set, these are

$$x(t)=-\frac{0.5t^2}{t+1}.$$