What smooth surfaces can be constructed from **total degree 2 splines?**

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https://www.cise.ufl.edu/~jorg/

Lower Bound Results

Degree estimates for C^k-piecewise polynomial subdivision surfaces

Hartmut Prautzsch[a] and Ulrich Reif[b,*]

C2 subdivision requires polynomial pieces of degree 2*3

" On the Complexity of Smooth Spline Surfaces from Quad Meshes "
Jörg Peters and Jianhua Fan
Computer Aided Geometric Design (CAGD) vol. 27, no. 1, pp. 96-105, **2010**

" Least Degree G^1-refinable Multi-sided Surfaces Suitable for Inclusion into C^1 bi-2 Splines "
Kestutis Karciauskas , Jörg Peters
CAD , **2020**

" A sharp degree bound on G^2 refinable multi-sided surfaces "
Kestutis Karciauskas , Jörg Peters
Solid and Physical Modeling , **2020** talk

Rich space of quadratic C1 splines!

+ C1 with constant 2nd derivatives:
  Powell-Sabin Splines, Zwart-Powell Splines

  Dierckx et al. (1992,7) Manni ,Sablonniere (2007), Speleers et al. (2006), Cohen et al. (2013); Windmolders and Dierckx (1999, 2000); Speleers et al. (2007, 2012, 2013); da Veiga et al. (2015), Lyche and Muntingh (2019)

+ exact-parametrizable planar slices (conics)
  (no numerical ambiguity → occluding contours/silhouettes Capouellez Sig23)

– shape deficient (not quite rich enough)

C1 splines use quadrilateral configurations

+ C1 splines with constant 2nd derivatives
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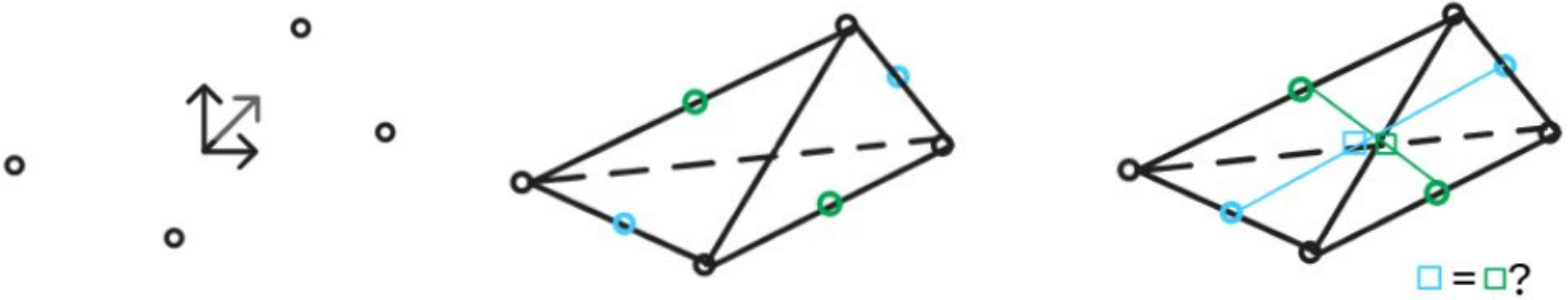

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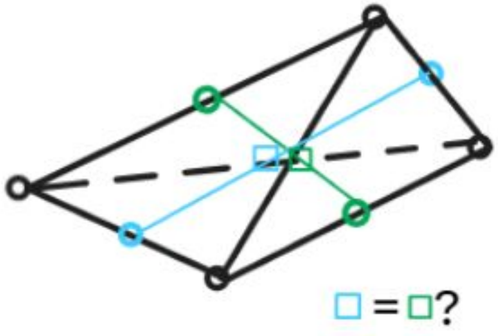

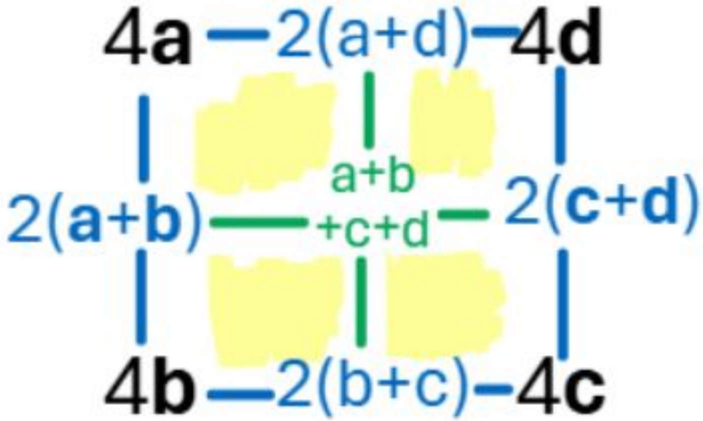

bi-linear

C1 splines use quadrilateral configurations

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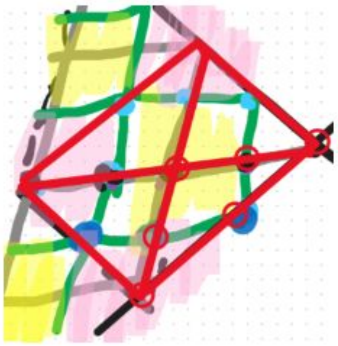
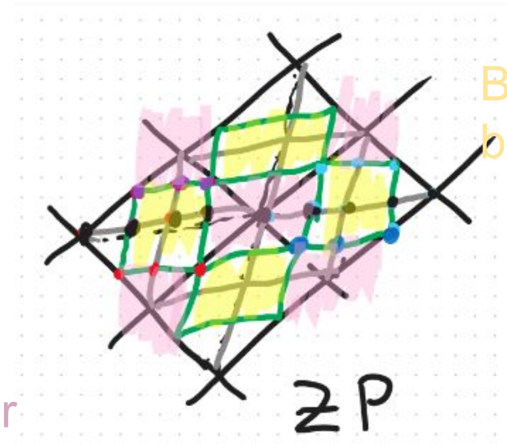
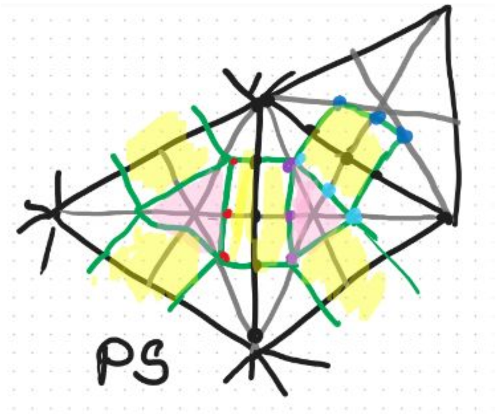


Stiff, planar



Bend, bilinear



C1 splines use quadrilateral configurations

+ C1 splines with constant 2nd derivatives
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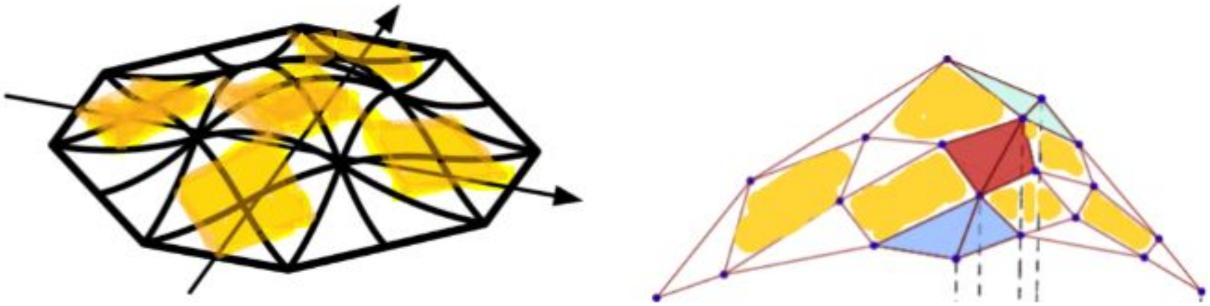
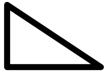


(a) ZP spline   (b) PS spline

Can total degree 2 splines model free-form surfaces?

1. Can Powell-Sabin (PS) or Zwart-Powell (ZP) constructions model smooth general free-form surfaces?

2. Can total degree 2 pieces (quadratics) model smooth general free-form surfaces? If so, how? (local?)

Assumptions: degree 2 splines
- flat, straight edge, domain triangles  → R^3
- G1 (C1) joined:

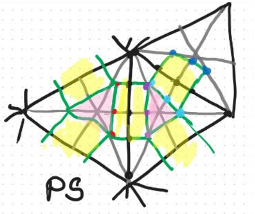
Lemma 1 (G^1 constraints, DeRose (1990); Peters (1990)). *Two polynomial pieces* $\mathbf{p}, \mathbf{q} : (u, v) \to \mathbb{R}^3$ *join* G^1 *along the common boundary* $\mathbf{p}(u, 0) = \mathbf{q}(u, 0), u \in [0..1]$ *if and only if there exist polynomials* $\lambda, \mu, \nu : [0..1] \to \mathbb{R}$, $\mu\nu > 0$, *so that*

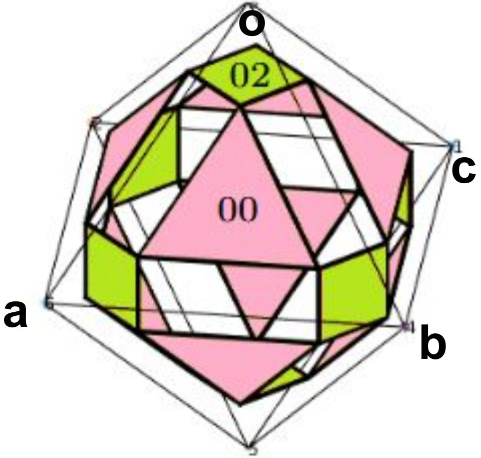
$$\lambda(u)\ \partial_u \mathbf{p}(u,0) = \mu(u)\ \partial_v \mathbf{p}(u,0) + \nu(u)\ \partial_v \mathbf{q}(u,0), \quad u \in [0..1]. \tag{G1}$$

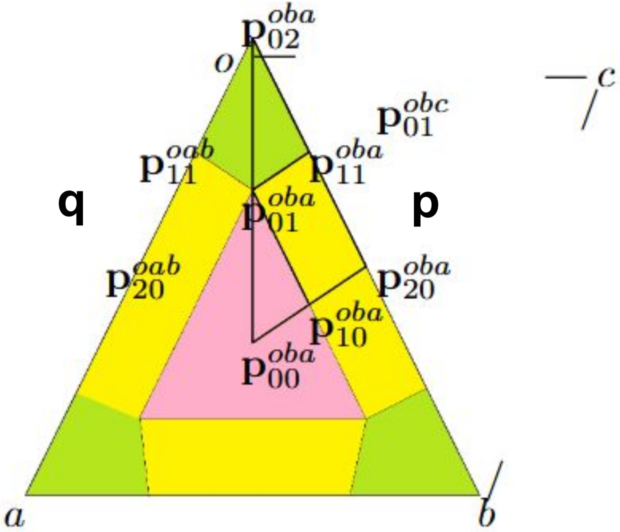
Can total degree 2 splines model free-form surfaces?

- A symmetric counterexample to free-form Powell-Sabin constructions
- Shape constraints for piecewise quadratic G1 surfaces of genus other than 1
- Special quadratic surface constructions

A symmetric counterexample to free-form Powell-Sabin constructions



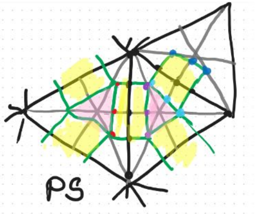


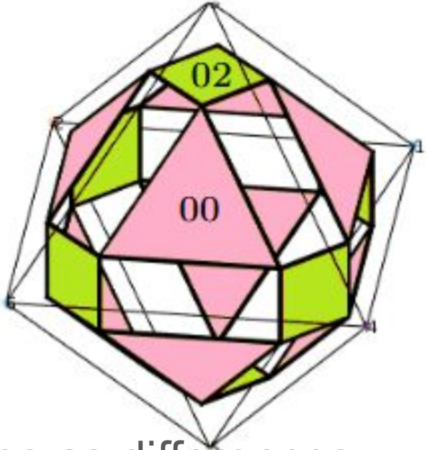


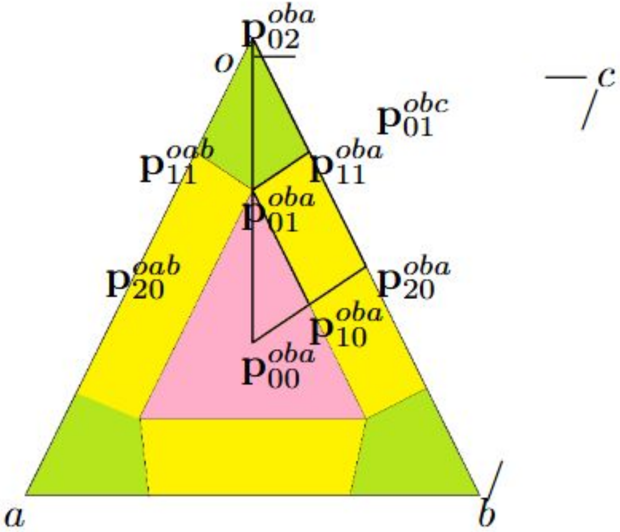
$$\partial_v \mathbf{p}(u,0) + \partial_v \mathbf{q}(u,0) = b(u)\partial_u \mathbf{p}(u,0), \quad u \in [0..1], \quad b : \mathbb{R} \to \mathbb{R}. \qquad (G^1_{sym})$$

1 1 1 → degree(b)=0 for PS spline

A symmetric counterexample to free-form Powell-Sabin constructions





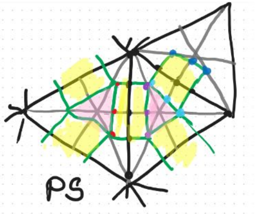


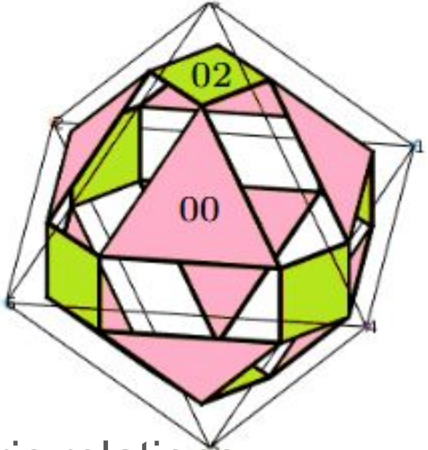
Express derivatives as differences

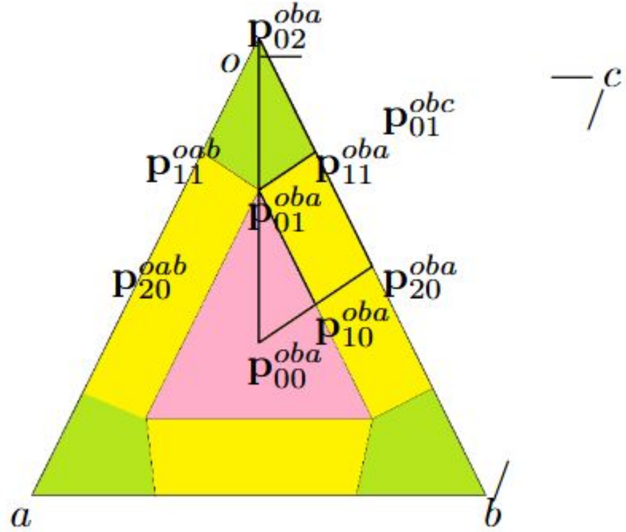
$$\partial_v \mathbf{p}(u,0) = 2(1-u)(\mathbf{p}_{11} - \mathbf{p}_{02}) + 2u(\mathbf{p}_{10} - \mathbf{p}_{01}),$$
$$\partial_v \mathbf{q}(u,0) = 2(1-u)(\mathbf{q}_{11} - \mathbf{q}_{02}) + 2u(\mathbf{q}_{10} - \mathbf{q}_{01}),$$
$$\partial_u \mathbf{p}(u,0) = 2(1-u)(\mathbf{p}_{01} - \mathbf{p}_{02}) + 2u(\mathbf{p}_{00} - \mathbf{p}_{01}).$$

$$\begin{bmatrix} \mathbf{p}_{11}-\mathbf{p}_{02} \\ \mathbf{p}_{10}-\mathbf{p}_{01} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_{11}-\mathbf{q}_{02} \\ \mathbf{q}_{10}-\mathbf{q}_{01} \end{bmatrix} = \beta \begin{bmatrix} \mathbf{p}_{01}-\mathbf{p}_{02} \\ \mathbf{p}_{00}-\mathbf{p}_{01} \end{bmatrix}$$

A symmetric counterexample to free-form Powell-Sabin constructions







Compute geometric relations:

$$\mathbf{p}_{11} - \mathbf{p}_{02} + \mathbf{q}_{11} - \mathbf{q}_{02} = (1 + c_n)(\mathbf{p}_{01} - \mathbf{p}_{02}), \quad c_n := \cos(2\pi/n)$$

$$\mathbf{p}_{10} - \mathbf{p}_{01} + \mathbf{q}_{10} - \mathbf{q}_{01} = (1 - c_3)(\mathbf{p}_{00} - \mathbf{p}_{01}).$$

$$1 + c_4 = 1 \neq \tfrac{3}{2} = (1 - c_3)$$

A symmetric counterexample to free-form Powell-Sabin constructions

$$1 + c_4 = 1 \neq \frac{3}{2} = (1 - c_3)$$

Can total degree 2 splines model free-form surfaces?

- A symmetric counterexample to free-form Powell-Sabin constructions
- Shape constraints for piecewise quadratic G1 surfaces of genus other than 1
- Special quadratic surface constructions

G1 splines

Lemma 1 (G^1 constraints, DeRose (1990); Peters (1990)). *Two polynomial pieces* $\mathbf{p}, \mathbf{q} : (u, v) \to \mathbb{R}^3$ *join* G^1 *along the common boundary* $\mathbf{p}(u, 0) = \mathbf{q}(u, 0), u \in [0..1]$ *if and only if there exist polynomials* $\lambda, \mu, \nu : [0..1] \to \mathbb{R}$, $\mu\nu > 0$, *so that*

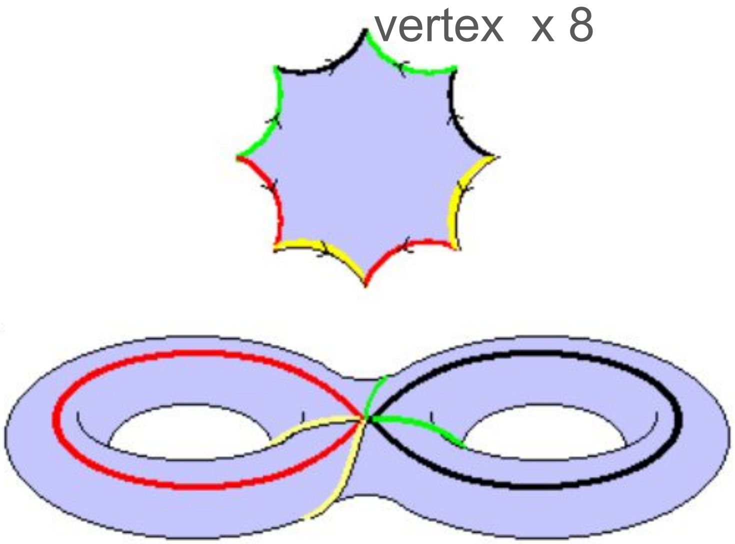
$$\lambda(u)\, \partial_u \mathbf{p}(u, 0) = \mu(u)\, \partial_v \mathbf{p}(u, 0) + \nu(u)\, \partial_v \mathbf{q}(u, 0), \quad u \in [0..1]. \tag{G1}$$

If λ, μ and ν are all constant

Observation 1. *Given a point on the* G^1 *surface surrounded without gap or overlap by surface pieces* $\mathbf{p}^i$, $i = 1, \ldots, n$. *If, for each pair* $\mathbf{p}^i, \mathbf{p}^{i+1}$, *the reparameterization is a single affine map then the domains of the* $\mathbf{p}^i$ *can be piecewise-affinely embedded into* $\mathbb{R}^2$ *to surround the pre-image of the point without gap and overlap.*

Shape constraints for piecewise quadratic G1 surfaces of genus other than 1

Lemma 5. *Let M be a closed compact surface constructed by enforcing (G1) of Lemma 1 between the pieces. If M has genus other than 1 then at least one pair* $\mathbf{p}, \mathbf{q}$ *has at least one of* λ, μ, ν *not constant.*



All G1 constraints are affine
= all λ, μ, ν are constant

→

Sum of vertex angles = 2 pi

→ Polygon (fundamental domain) is rectangle

→ genus 1

Shape constraints for piecewise quadratic G1 surfaces of genus other than 1

Lemma 5. *Let M be a closed compact surface constructed by enforcing (G1) of Lemma 1 between the pieces. If M has genus other than 1 then at least one pair $\mathbf{p}, \mathbf{q}$ has at least one of λ, μ, ν not constant.*

$$\bar{\Gamma}^i_{kl} = \frac{\partial \bar{\mathbf{u}}^i}{\partial \mathbf{u}^m} \frac{\partial \mathbf{u}^n}{\partial \bar{\mathbf{u}}^k} \frac{\partial \mathbf{u}^p}{\partial \bar{\mathbf{u}}^l} \Gamma^m_{np} + \frac{\partial^2 \mathbf{u}^m}{\partial \bar{\mathbf{u}}^k \partial \bar{\mathbf{u}}^l} \frac{\partial \bar{\mathbf{u}}^i}{\partial \mathbf{u}^m}.$$

Christoffel symbols → Xianfeng David Gu

choose $\Gamma^{\mu}_{\nu\sigma} = 0$ affine ρ, $\frac{\partial^2 \mathbf{u}^m}{\partial \bar{\mathbf{u}}^k \partial \bar{\mathbf{u}}^l} = 0$

→ K=0 in Gauss Bonnet $\int_{\tilde{M}} K dA = 2\pi\chi(\tilde{M})$

→ 0 = 2-2g

Shape constraints for piecewise quadratic G1 surfaces of genus other than 1

any reasonable G^1 construction should allow for (not rule out) construction symmetry of the form $\mu = \nu$

$$\begin{bmatrix} b(0)\partial_u \mathbf{q}(0,0) \\ b(1)\partial_u \mathbf{q}(1,0) \end{bmatrix} = \begin{bmatrix} \partial_v \mathbf{p}(0,0) \\ \partial_v \mathbf{p}(1,0) \end{bmatrix} + \begin{bmatrix} \partial_v \mathbf{q}(0,0) \\ \partial_v \mathbf{q}(1,0) \end{bmatrix}, \quad b(0) := \frac{\lambda(0)}{\nu(0)},\ b(1) := \frac{\lambda(1)}{\nu(1)}$$

Theorem 1 (shape constraints). *Consider an endpoint* $\mathbf{e}$ *of a boundary between a pair* $\mathbf{p}, \mathbf{q}$ *of total degree 2 surface pieces satisfying (G1) with at least one non-constant triple* λ, μ, ν *with* $\mu = \nu$ *and* $\mu \neq 0$ *at* $\mathbf{e}$. *Then, at* $\mathbf{e}$, *the surface*

- *either has a flat region,*

or else, and only if the number of patches joining at the vertex is an even $2n$,

- *an oscillation with* n *maxima and* n *minima.*

Shape constraints for piecewise quadratic G1 surfaces of genus other than 1

Theorem 1 (shape constraints). *Consider an endpoint* $\mathbf{e}$ *of a boundary between a pair* $\mathbf{p}, \mathbf{q}$ *of total degree 2 surface pieces satisfying (G1) with at least one non-constant triple* λ, μ, ν *with* $\mu = \nu$ *and* $\mu \neq 0$ *at* $\mathbf{e}$*. Then, at* $\mathbf{e}$*, the surface*

- *either has a flat region,*

or else, and only if the number of patches joining at the vertex is an even $2n$*,*

- *an oscillation with* n *maxima and* n *minima.*

$$0 = \det[\mathbf{q}_u, \mathbf{p}_v, \mathbf{q}_v](u), \qquad \mathbf{q}_u(u) := (\partial_u \mathbf{p})(u,0),\ \mathbf{p}_v(u) := (\partial_v \mathbf{p})(u,0),\ \mathbf{q}_v(u) := (\partial_v \mathbf{q})(u,0)$$

Coefficient 1,2 of det polynomial of degree 3 set to zero

$$\mathbf{n}(0) \cdot \mathbf{q}_u(1) = 0 \text{ and } \mathbf{n}(1) \cdot \mathbf{q}_u(0) = 0$$

$\mathbf{q}_u(0) \times \mathbf{q}_u(1) \neq 0$ → flat

$$0 = \mathbf{n}(0) \cdot ((\partial_u \partial_v \mathbf{p})(0,0) + (\partial_u \partial_v \mathbf{q})(0,0))$$

→ flat or oscillating 2n times

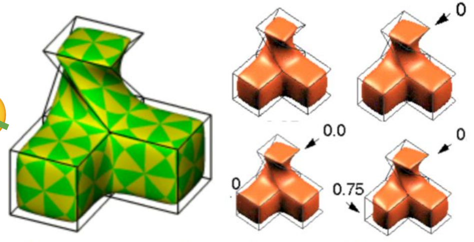
Can total degree 2 splines model free-form surfaces?

- ➢ A symmetric counterexample to free-form Powell-Sabin constructions ✔
- ➢ Shape constraints for piecewise quadratic G1 surfaces of genus other than 1✔
- ➢ Special quadratic surface constructions

Special quadratic surface constructions

- Constructions restricted to domains of genus 1 ☹️
- Constructions leaving out cut points ☹️
- Constructions using non-standard domains 😐
- Constructions as level sets of trivariate quadratics 😐
- Constructions with a singular parameterization (are flat) ☹️
- Constructions using infinitely many pieces (simplest subdivision) 😐
- Constructions with additional degree 3 pieces ☹️
- Constructions enforcing flat regions 😐
- Constructions enforcing saddles of high order (too many) ☹️

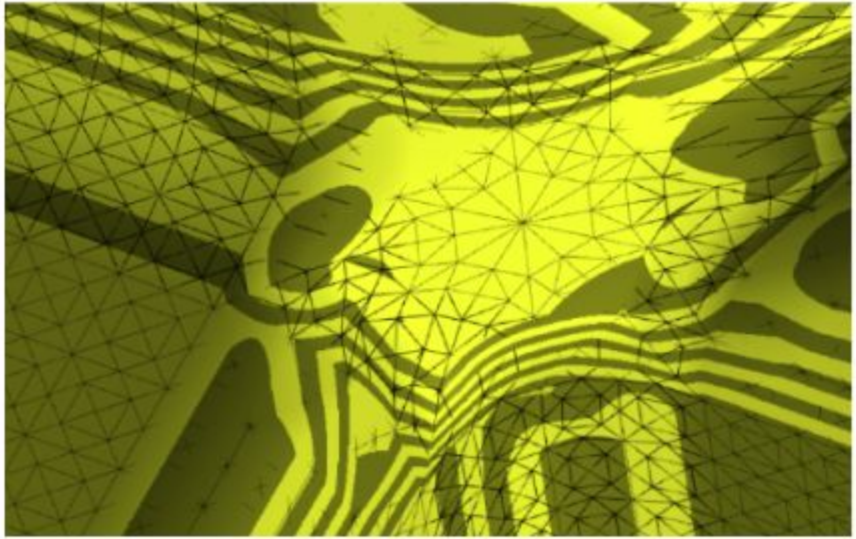
(a) Simplest Subdivision



(b) Valence (2×) $n \in \{3, 4, 5, 6\}$ (c) Blend ratios in $[0..1]$ for semi-smooth creases

Special quadratic surface constructions

Constructions enforcing flat regions



Conclusion

- No piecewise quadratic G1 surfaces of genus ≠ 1
- except for the intentional introduction of flat spots