Notation

Jorg Peters

We could, of course, use any notation we want;
do not laugh at notations;
invent them, they are powerful.
In fact, mathematics is, to a large extent,
invention of better notations.

Richard P. Feynman

BB-form(ulas)

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The derivative $\partial \mathbf{p}_n$ of a polynomial $\mathbf{p}_n$ in Bernstein form must be writable as a polynomial of one degree less in Bernstein form:

Differentiation = Differencing coefficients

$$\partial\Big(\sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}\Big) = \sum_{i=0}^{n-1} \mathbf{d}_{n-1-i,i} b_{n-1-i,i} \qquad \mathbf{d}_{n-i,i} := n(\mathbf{c}_{n-i-1,i+1} - \mathbf{c}_{n-i,i}).$$

$$\sum_{i=0}^{n_1} \mathbf{c}^1_{n_1-i,i} b_{n_1-i,i} * \sum_{i=0}^{n_2} \mathbf{c}^2_{n_2-i,i} b_{n_2-i,i} = \sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}$$

where $n = n_1 + n_2$, and

$$\mathbf{c}_{n-i,i} = \sum_{i_1+i_2=i} \frac{\binom{n_1}{i_1}\binom{n_2}{i_2}}{\binom{n}{i}} \mathbf{c}^1_{n-i_1,i_1} \mathbf{c}^2_{n-i_2,i_2}.$$

Multiplication = Collecting coefficients with equal index sums

$$\int_0^1 \sum \mathbf{c}_{n-i,i} b_{n-i,i} du = \sum \mathbf{c}_{n-i,i}/(n+1).$$

Integration = Summing coefficients

Total Degree BB-form

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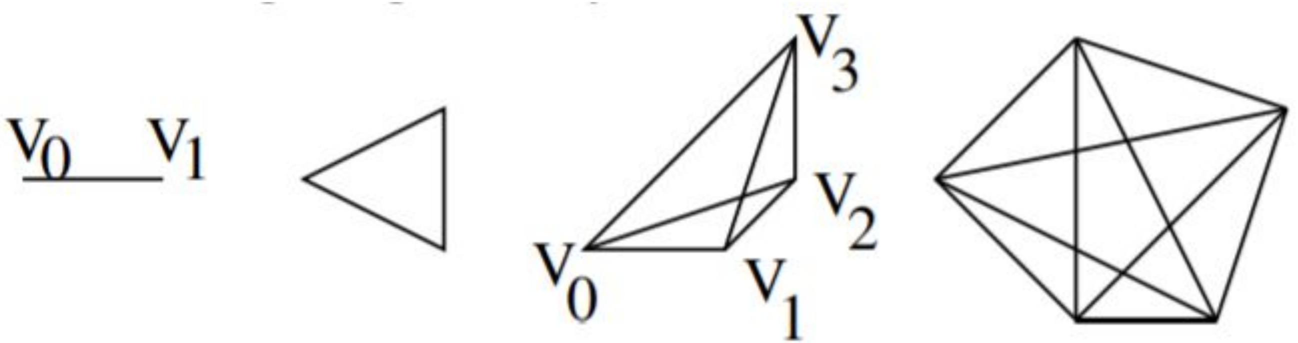
$\mathbf{v}_i \in \mathbb{R}^m$, and $\mathbf{V} := [\mathbf{v}_0 \mathbf{v}_1 \cdots \mathbf{v}_m]$



$\xi_{\nu_i} : \mathbb{R}^m \to \mathbb{R}, \quad \xi_{\nu_i}(\mathbf{v}_j) = e_i(j).$

Total Degree BB-form

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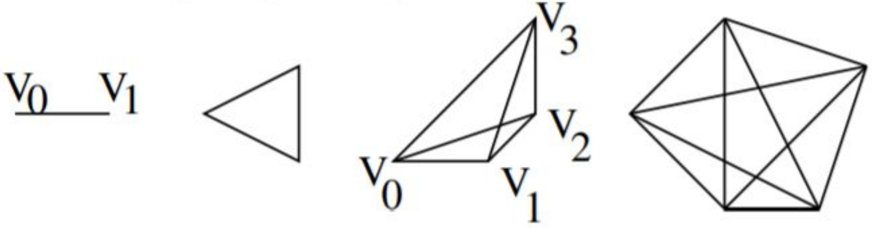

$$\xi_{\nu_i} : \mathbb{R}^m \to \mathbb{R}, \quad \xi_{\nu_i}(\mathbf{v}_j) = e_i(j).$$

$$\mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ i.e. } \mathbf{V} := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

has barycentric coordinate functions (expressed in Cartesian x and y)

$$\xi_0(x,y) = y,$$
$$\xi_1(x,y) = 1 - x - y,$$
$$\xi_2(x,y) = x.$$

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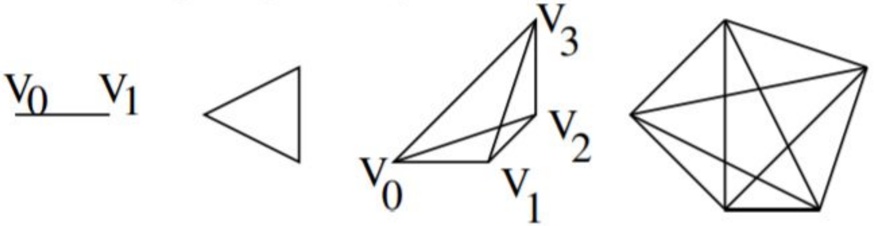

$$\xi_{\nu_i} : \mathbb{R}^m \to \mathbb{R}, \quad \xi_{\nu_i}(\mathbf{v}_j) = e_i(j).$$

A polynomial p of total degree d in m variables and with coefficients $c(\mathcal{V}\mathbf{i})$ is in Bernstein-Bézier form, or, short, *BB form*, with respect to the simplex $\mathcal{V}$ if

$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

The vector of indices fat i = **i** has entries i_0, i_1, ... i_m where m is the number of variables (mnemonic hint for the use of m: multi-variate) It helps to read everything for m=1 since you already know the 1-variable case very well, where we have two indices i_0 and i_1 that add to the degree , ie **|i|** := i_0+i_1 = d (mnemonic hint for d: degree) The barycentric coordinate functions in 1 variable are xi_0 := (1-t) and xi_1 := t and the basis in the BB-form is binomial{d}{k} (xi_0)^{d-k} + (xi_1)^k.

Total Degree BB-form

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$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

A *d-control point* is a map from an expression to a scalar: $\mathcal{V}\mathbf{i} \to \mathbb{R}$.We associate

$$c(\mathcal{V}\mathbf{i}) \quad \text{with the Greville abscissa} \quad \frac{1}{|\mathbf{i}|}\mathbf{V}\mathbf{i}.$$

The *control net* of the points

$$(\frac{1}{|\mathbf{i}|}\mathbf{V}\mathbf{i}, c(\mathcal{V}\mathbf{i})),$$

Total Degree BB-form

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$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

$$\begin{aligned}
&c(300) = c(201) = c(102) = 0,\\
&c(210) = c(111) = c(012) = 2,\\
&c(120) = c(021) = 4,\\
&c(030) = 14,\ c(003) = 1,
\end{aligned}$$

$$\begin{matrix}
0 & & & \\
2 & 0 & & \\
4 & 2 & 0 & \\
14 & 4 & 2 & 1
\end{matrix}$$

Total Degree BB-form

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$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

DeCasteljau

Input: $\omega := \xi(\mathbf{x})$ and $c(\mathcal{V}\mathbf{i}), |\mathbf{i}| = d$.

Output: $p(\mathbf{x}) = c(0)$ and $c(\mathcal{V}^j\mathbf{i}), j = 0, \ldots, m$.

for $\ell = 1..d$

  for $|\mathbf{j}| = d - \ell$

    $c(\mathcal{V}\mathbf{j}) = \sum_{i=0}^{m} \omega_i \, c(\nu\mathbf{j} + \nu_i)$

0			
2	0		
4	2	0	
14	4	2	1

Total Degree BB-form

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$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

$$p_{\mathcal{V}^i} := \sum_{|\mathbf{i}|=d} (\xi_{\mathcal{V}^i})^{\mathbf{i}} \binom{d}{\mathbf{i}} c(\mathcal{V}^i \mathbf{i}),$$

Each $p_{\mathcal{V}^i}$ represents p over a

subsimplex $\mathcal{V}^i := [\nu_0, \ldots, \nu_{i-1}, \mathcal{V}\omega, \nu_{i+1}, \ldots, \nu_m]$.

0			
2	0		
4	2	0	
14	4	2	1

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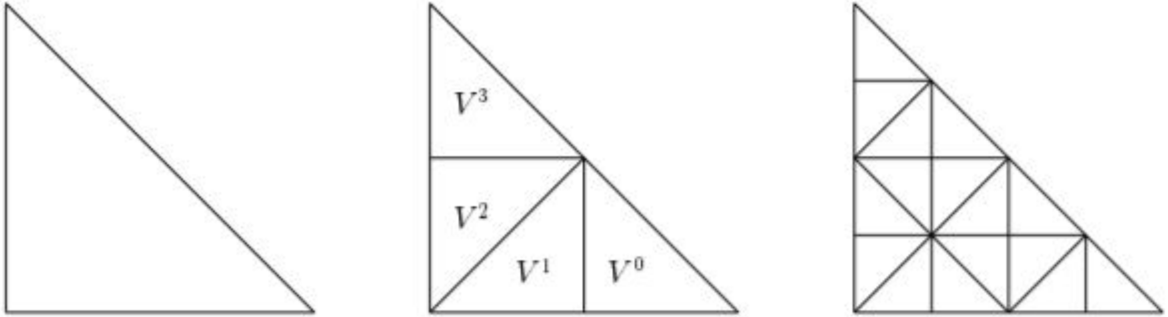
$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$



0			
0	2		
0	2	4	
0	2	4	14

$\Rightarrow .. \Rightarrow$

0						
	1					
0		2				
	1		4			
0		2		6		
	1		3		9	
0		2		4		14

$\Rightarrow .. \Rightarrow$

0						
0	1					
0	1	2				
0	1	2	4			
0	1	2	4	6		
0	1	2	4	6	9	
0	1	2	4	6	9	14

Total Degree BB-form

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$$p = \sum_{|\mathbf{i}|=d} c(\mathcal{V}\mathbf{i}) B_{\mathcal{V}\mathbf{i}}, \qquad B_{\mathcal{V}\mathbf{i}} := \binom{d}{\mathbf{i}} \xi^{\mathbf{i}}.$$

$$D_{\mathbf{v}_i - \mathbf{v}_j} \xi_k = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = j \\ 0 & \text{else.} \end{cases}$$

$$\begin{matrix} 0 & & & \\ 2 & 0 & & \\ 4 & 2 & 0 & \\ 14 & 4 & 2 & 1 \end{matrix}$$

$$D_{\mathbf{v}_2 - \mathbf{v}_1} p \sim \begin{matrix} -6 & & \\ -6 & -6 & \\ -30 & -6 & -3 \end{matrix}$$

Total Degree BB-form

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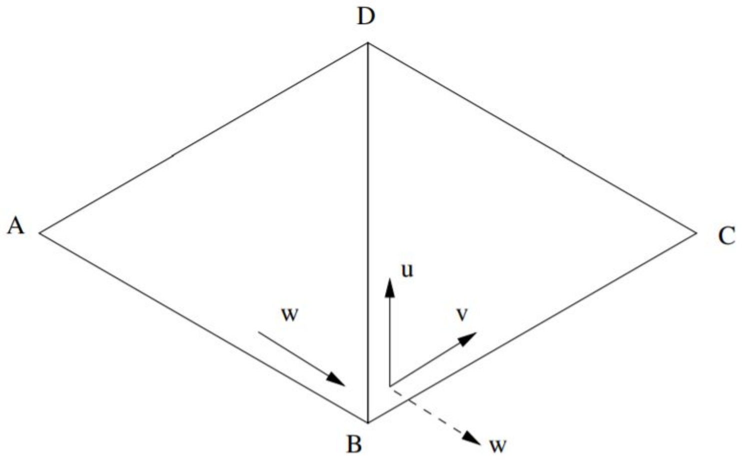
Example: If $m = 2$ and $\mathbf{v}_0 = \sum_{i=1}^3 \omega_i \mathbf{v}_i$ then

$$D_{\mathbf{v}_1-\mathbf{v}_0} = D_{\mathbf{v}_1-\sum_{i=1}^3 \omega_i \mathbf{v}_i} = -\omega_2 D_{\mathbf{v}_2-\mathbf{v}_1} - \omega_3 D_{\mathbf{v}_3-\mathbf{v}_1}$$



$$(\mathbf{v}_0 - \mathbf{v}_1) + (\mathbf{v}_2 - \mathbf{v}_1) = 2\cos\tfrac{2\pi}{n}(\mathbf{v}_3 - \mathbf{v}_1)$$

$$D_{\mathbf{v}_1-\mathbf{v}_0} = D_{\mathbf{v}_2-\mathbf{v}_1} - 2\cos\frac{2\pi}{n} D_{\mathbf{v}_3-\mathbf{v}_1}.$$

Total Degree BB-form

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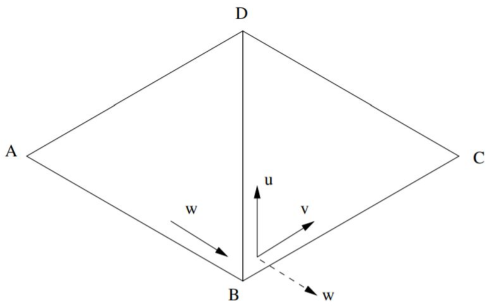

$$D_{\mathbf{v}_1-\mathbf{v}_0} = D_{\mathbf{v}_2-\mathbf{v}_1} - 2\cos\frac{2\pi}{n} D_{\mathbf{v}_3-\mathbf{v}_1}.$$

$$\begin{aligned}
&c((d-i)\nu_1 + i\nu_3) - c(\nu_0 + (d-i-1)\nu_1 + i\nu_3) \\
&= c(\nu_2 + (d-i-1)\nu_1 + i\nu_3) - c((d-i)\nu_1 + i\nu_3) \\
&- 2\cos\frac{2\pi}{n}\left(c((d-i-1)\nu_1 + (i+1)\nu_3) - c((d-i)\nu_1 + i\nu_3)\right)
\end{aligned}$$

Total Degree BB-form

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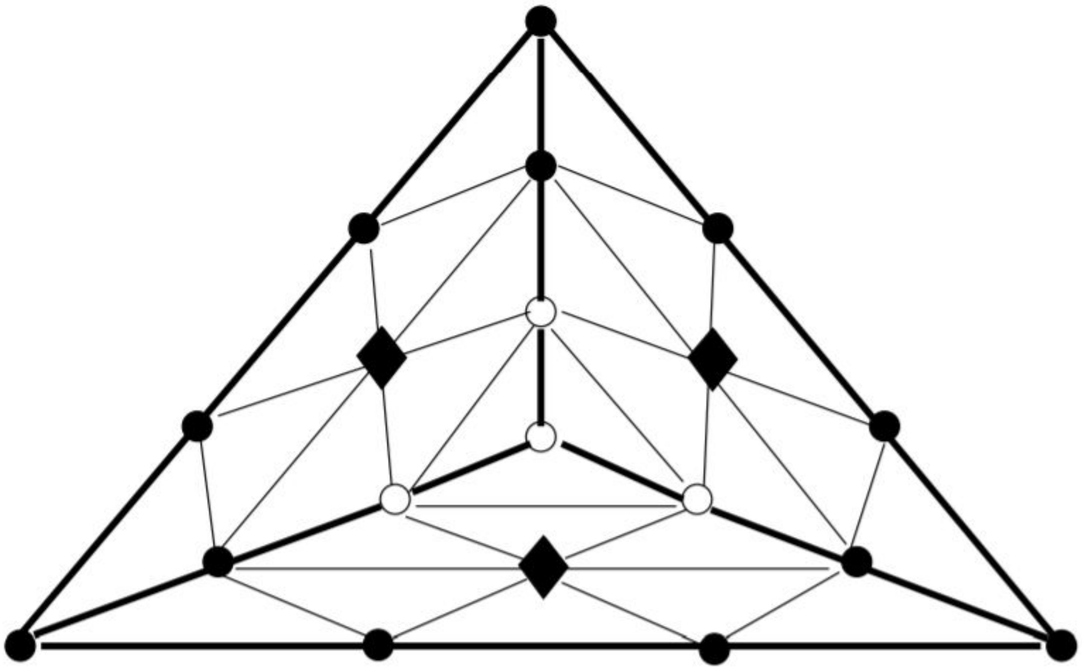

Figure D.2: The Clough-Tocher interpolant.