Problem 1  (10 points)

A size of a set is the number of elements in a set. $S$ & $T$ are two set of sets (a set whose members are sets themselves). $T$ is cover of $S$ iff $T$ is a subset of $S$ and if $\cup T = \cup S$. $T$ is a minimum set cover when the size of $T$ is minimum.

You create the minimum set cover $T$ in iterations. Initially, $T$ is the null set. In each iteration, you add one member of $S$ (which itself is a set) to $T$ until all the elements in the members of $S$ are “covered”. The way you make the choice is by determining which member of $S$ has the maximum number of elements still not covered by $T$.

Like, for $S = \{ \{1,2,3,4,5,6\}, \{3,4,5,6,7\}, \{5,6,7,8\}, \{7,8,9,10\} \}$,
$T = \{ \{1,2,3,4,5,6\}, \{7,8,9,10\} \}$
Observe that $\cup T = \cup S = \{1,2,3,4,5,6,7,8,9,10\}$

Does this Greedy algorithm always yield an optimal solution? If yes, give a proof why it does. Otherwise, provide a counterexample.

Problem 2  (15 points)

Linda, the CISE secretary handles the E305 conference room reservation, and the professors send her the reservation requests in the format of <meetingStartTime, meetingEndTime>. Linda does the reservation scheduling based upon the maximum number of meetings, not on how long the conference room has been utilized. If two schedules have same number of meetings, she can break ties arbitrarily. Suppose by Monday 5 PM Linda has a list of requests for Tuesday (with distinct start/end times), with Dr. Ranka’s request being the one that finishes first among all requested meetings. Now, can there be ANY possible scenario where Dr. Ranka still MUST be refused to ensure optimality (i.e., can it happen that accommodating Dr. Ranka’s meeting would cause a sub-optimal solution)? If yes, give an example. Otherwise, justify your choice.

Problem 3.  (15 points)

You are the Project Manager in a company where the length of working hours and the work start time is different for different employees. For example, person A works everyday from 8 AM – 10 AM, person B from 9 AM – 3 PM, person C from 1 PM – 10 PM etc. You want to assemble a workforce consisting of maximum number of people with non-clashing working hours (if a person leaves at 1PM and another starts at 1PM, that’s not considered a clash). You use the following Greedy strategy:

- Choose the person (say person X) with least number of clashes.
- Eliminate people having working hour clashes with person X.
- Choose the next person having the least number of clashes with the remaining people (after choosing each person, you are also eliminating some), and so on.

Does this Greedy strategy necessarily yield the set of maximum people? If yes, prove it. Else, just give a counterexample.
Problem 4  (10 points)

You are given an undirected graph $G = (V, E)$ of distinct edge weights. Design an algorithm that takes $G$ as input, and removes a set of edges of minimum total weight, so that the output is a graph $G' = (V, E')$ which contains exactly one path between any two vertices (Observe that $G'$ has the same set of vertices as $G$).

Problem 5  (15 points)

A road network consists of:

- **Intersections** (crossing of two or more roads, or dead ends) and
- **Road segments** (portion of the road between a pair of intersections).

A traffic sergeant standing at an intersection can watch all the road segments (till the next intersection) in all the directions. You want to choose the minimum number of intersections so that placing traffic sergeants at those intersections will allow the monitoring of all the road segments.

You use the following greedy strategy repetitively:

- Choose the intersection having the max number of road segments connected to it.
- Eliminate the road segments connected to that intersection.

*(Observe that every time you are making the choice over a changed set of road segments.)*

Does this greedy strategy essentially yield a minimum number of intersections? If yes, prove it. Else, provide a counterexample.

Problem 6  (20 points)

You are going from Gainesville to Seattle via interstate highways. While traveling over interstate highways, there are two travel costs incurred: 1) fuel cost, which is proportional to the length of the interstate and 2) toll cost, which are collected at tollbooths. There is a tollbooth at every 1) Dead end of a highway and 2) junction (intersection) of two or more highways. In addition, there could be tollbooths on a single highway itself (no intersection). You can consider the exit & entry ramps as dead ends (which means you will have to pay for getting on or getting off a highway). You have a map of the highway network that gives all the information about the road network, tollbooths and payable tolls. See if you can modify the problem to use the Dijkstra’s algorithm for calculating the optimum travel cost from Gainesville to Seattle.

Problem 7  (15 points)

You are given a graph of distinct edge weights. Now, for every vertex, determine which is the lightest edge that is incident upon it. Create a set consisting of these lightest edges. What is the minimum size of this set? Will it necessarily be a subset of the MST? If yes, justify why. Otherwise, give a counterexample.