Problem 1 [Section 1.5 Exercise 14 (f)(g) (5 points)]
Use quantifiers and predicates with more than one variable to express these statements.
f) Some students in this class grew up in the same town as exactly one other student in this class.
g) Every student in this class has chatted with at least one other student in at least one chat group.
Solution
The answers to this exercise are not unique; there are many ways of expressing the same propositions symbolically. Our domain of discourse for persons here consists of people in this class. We need to make up a predicate in each case.
f) Let \( G(x, y) \) mean that persons \( x \) and \( y \) grew up in the same town. Then our statement is
\[
\exists x \exists y (x \neq y \land G(x, y) \land \forall z (G(x, z) \rightarrow (x = y \lor x = z))).
\]
g) Let \( C(x, y, z) \) mean that persons \( x \) and \( y \) have chatted with each other in chat group \( z \). Then our statement is
\[
\forall x \exists y \exists z (z = (x + y)/2).
\]
Problem 2 [Section 1.5 Exercise 28 (f)(h)(i)(j) (10 points)]
Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
f) \( \exists x \forall y (y \neq 0 \rightarrow xy = 1) \)
h) \( \exists x \exists y (x + 2y = 2 \land 2x + 4y = 5) \)
i) \( \forall x \forall y (x + y = 2 \land 2x - y = 1) \)
j) \( \forall x \forall y \exists z (z = (x + y)/2) \)
Solution
f) false (the reciprocal of \( y \) depends on \( y \) — there is not one \( x \) that works for all \( y \))
h) false (this system of equations is inconsistent)
i) false (this system has only one solution; if \( x = 0 \), for example, then no \( y \) satisfies \( y = 2 \land y = 1 \))
j) true (let \( z = (x + y)/2 \))
Problem 3 [Section 1.5 Exercise 30 (c)(d) (5 points)]
Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
c) \( \neg \exists y (Q(y) \land \forall x \lnot R(x, y)) \)
d) \( \neg \exists y (\exists x R(x, y) \lor \forall x S(x, y)) \)
Solution
We need to use the transformations shown in Table 2 of Section 1.4, replacing \( \neg \forall y \exists z \rightarrow \neg \exists z \rightarrow \forall y \exists z \) and replacing \( \neg \exists \rightarrow \forall \exists \). In other words, we push all the negation symbols inside the quantifiers, changing the sense of the quantifiers as we do so, because of the equivalences in Table 2 of Section 1.4. In addition, we need to use De Morgan’s laws (Section 1.3) to change the negation of a conjunction to the disjunction of the negations and to change the negation of a disjunction to the conjunction of the negations. We also use the fact that \( \neg \neg p = p \).
c) \( \forall y (\neg Q(y) \lor \exists x \lnot R(x, y)) \)
d) \( \forall y (\forall x \lnot R(x, y) \land \exists x \lnot S(x, y)) \)
Problem 4 [Section 1.6 Exercise 12 (10 points)]
Show that the argument form with premises \( (p \land t) \rightarrow (r \lor s) \), \( q \rightarrow (u \land t) \), \( u \rightarrow p \), and \( \neg s \) and conclusion \( q \rightarrow r \) is valid by first using Exercise 11 of Section 1.6 and then using the rules of inference from Table 1 of Section 1.6.
Exercise 11 (You can directly use the conclusion without proving it): Show that the argument form with premises \( p_1, p_2, \ldots, p_n \) and conclusion \( q \rightarrow r \) is valid if the argument form with premises \( p_1, p_2, \ldots, p_n, q \) and conclusion \( r \) is valid.
Solution
Applying Exercise 11, we want to show that the conclusion r follows from the five premises \((p \land t) \rightarrow (r \lor s)\), \(q \rightarrow (u \land t)\), \(u \rightarrow p\), \(\neg s\), and \(q\). From \(q\) and \(q \rightarrow (u \land t)\) we get \(u \land t\) by modus ponens. From there we get both \(u\) and \(t\) by simplification (and the commutative law). From \(u \land t\) we get \(p\) by modus ponens. From \(p\) and \(t\) we get \(p \land t\) by conjunction. From that and \((p \land t) \rightarrow (r \lor s)\) we get \(r \lor s\) by modus ponens. From that and \(\neg s\) we finally get \(r\) by disjunctive syllogism.

Problem 5 [Section 1.6 Exercise 15 (10 points)]
For each of these arguments determine whether the argument is correct or incorrect and briefly explain why (no more than two lines for each question).

a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
Solution a) Correct, using Universal instantiation and modus ponens

b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
Solution b) Invalid, fallacy of affirming the conclusion

c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
Solution c) Invalid; fallacy of denying the hypothesis

d) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.
Solution d) Correct, using Universal instantiation and modus tollens

Problem 6 [Section 1.6 Exercise 20 (10 points)]
Determine whether these are valid arguments.

a) If \(x\) is a positive real number, then \(x^2\) is a positive real number. Therefore, if \(a^2\) is positive, where \(a\) is a real number, then \(a\) is a positive real number.
Solution a) This is invalid. It is the fallacy of affirming the conclusion. Letting \(a = -2\) provides a counter example.

b) If \(x^2 \neq 0\), where \(x\) is a real number, then \(x \neq 0\). Let \(a\) be a real number with \(a^2 \neq 0\); then \(a \neq 0\).
Solution b) This is valid; it is modus ponens.

Problem 7 [Section 1.7 Exercise 7 (10 points)]
Use a direct proof to show that every odd integer is the difference of two squares.
Solution
If \(n\) is odd, we can write \(n = 2k + 1\), some integer \(k\). Then, taking the difference, \((k+1)^2 - k^2 = 2k + 1 = n\). Thus, every odd integer is the difference of two squares.

Problem 8 [Section 1.7 Exercise 18 (10 points)]
Prove that if \(n\) is an integer and \(n^3 + 5\) is odd, then \(n\) is even using

a) a proof by contraposition
b) a proof by contradiction
Solution
a) We must prove the contrapositive: If \(n\) is odd, then \(3n + 2\) is odd. Assume that \(n\) is odd. Then we can written \(= 2k + 1\) for some integer \(k\). Then \(3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 1) + 1\). Thus \(3n + 2\) is two times some integer plus 1, so it is odd.

b) Suppose that \(3n + 2\) is even and that \(n\) is odd. Since \(3n + 2\) is even, so is \(3n\). If we add subtract an odd
number from an even number, we get an odd number, so $3n-n=2n$ is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

**Problem 9 [Section 1.7 Exercise 22 (10 points)]**
Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

**Solution**
We give a proof by contradiction. Suppose that we don’t get a pair of blue socks or a pair of black socks. Then we drew at most one of each color. This accounts for only two socks. But we are drawing three socks. Therefore our supposition that we did not get a pair of blue socks or a pair of black socks is incorrect and our proof is complete.

**Problem 10 [Section 1.8 Exercise 10 (10 points)]**
Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?

**Solution**
The only perfect squares that differ by 1 are 0 and 1. Therefore these two consecutive integers cannot both be perfect squares. This is a non-constructive proof—we do not know which of them meets the requirement.
(In fact, a computer algebra system will tell us that neither of them is a perfect square.)

**Problem 11 [Section 1.8 Exercise 12 (10 points)]**
Show that the product of two of the three numbers $65^{1000} - 8^2001 + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, $24^{4493} - 5^8192 + 7^{1777}$ is nonnegative. Is your proof constructive or nonconstructive?

**Solution**
Of these three numbers, at least two must have the same sign (both positive or both negative), since there are only two signs. (It is conceivable that some of them are zero, but we view zero as positive for the purposes of this problem.) The product of two with the same sign is nonnegative. This was a nonconstructive proof, since we have not identified which product is nonnegative. (In fact, a computer algebra system will tell us that all three are positive, so all three products are positive.)

**Problem Bonus Question [Section 1.8 Exercise 28 (1 point for course total)]**
Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.

**Solution**
Clearly only the last two digits of $n$ contribute to the last two digits of $n^2$. So we can compute $0^2$, $1^2$, $2^2$, $3^2$, ..., $99^2$, and record the last two digits, omitting repetitions. We obtain 00, 01, 04, 09, 16, 25, 36, 49, 64, 81, 21, 44, 69, 96, 56, 89, 24, 61, 41, 84, 29, 76. From that point on, the list repeats in reverse order (as we take the squares from 25$^2$ to 49$^2$, and then it all repeats again as we take the squares from 50$^2$ to 99$^2$). The reason for these last two statements is that $(50-n)^2 = 2500 - 100n + n^2$, so $(50-n)^2$ and $n^2$ have the same two final digits, and $(50+n)^2 = 2500 + 100n + n^2$, so $(50+n)^2$ and $n^2$ have the same two final digits. Thus our list (which contains 22 numbers) is complete.