Backtracking And Branch And Bound
Subset & Permutation Problems

• Subset problem of size $n$.
  ▪ Nonsystematic search of the space for the answer takes $O(p2^n)$ time, where $p$ is the time needed to evaluate each member of the solution space.

• Permutation problem of size $n$.
  ▪ Nonsystematic search of the space for the answer takes $O(pn!)$ time, where $p$ is the time needed to evaluate each member of the solution space.

• Backtracking and branch and bound perform a systematic search; often taking much less time than taken by a nonsystematic search.
Tree Organization Of Solution Space

- Set up a tree structure such that the leaves represent members of the solution space.
- For a size $n$ subset problem, this tree structure has $2^n$ leaves.
- For a size $n$ permutation problem, this tree structure has $n!$ leaves.
- The tree structure is too big to store in memory; it also takes too much time to create the tree structure.
- Portions of the tree structure are created by the backtracking and branch and bound algorithms as needed.
Subset Problem

• Use a full binary tree that has $2^n$ leaves.
• At level $i$ the members of the solution space are partitioned by their $x_i$ values.
• Members with $x_i = 1$ are in the left subtree.
• Members with $x_i = 0$ are in the right subtree.
• Could exchange roles of left and right subtree.
Subset Tree For $n = 4$

$x_1 = 1$

$x_2 = 1$

$x_3 = 1$

$x_4 = 1$

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

$x_1 = 0$

$x_2 = 1$

$x_3 = 0$

$x_4 = 1$

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

$1110$

$1011$

$0111$

$0001$
Permutation Problem

- Use a tree that has $n!$ leaves.
- At level $i$ the members of the solution space are partitioned by their $x_i$ values.
- Members (if any) with $x_i = 1$ are in the first subtree.
- Members (if any) with $x_i = 2$ are in the next subtree.
- And so on.
Permutation Tree For $n = 3$

123 132 213 231 312 321
Backtracking

- Search the solution space tree in a depth-first manner.
- May be done recursively or use a stack to retain the path from the root to the current node in the tree.
- The solution space tree exists only in your mind, not in the computer.
Backtracking Depth-First Search
Backtracking Depth-First Search

\[ x_1 = 1 \]
\[ x_1 = 0 \]
\[ x_2 = 1 \]
\[ x_2 = 0 \]
Backtracking Depth-First Search

$x_1 = 1$

$x_2 = 1$

$x_1 = 0$

$x_2 = 0$

$x_2 = 1$

$x_2 = 0$
Backtracking Depth-First Search
Backtracking Depth-First Search
O(2^n) Subset Sum & Bounding Functions

{10, 5, 2, 1}, c = 14

Each forward and backward move takes $O(1)$ time.
Bounding Functions

• When a node that represents a subset whose sum equals the desired sum $c$, terminate.

• When a node that represents a subset whose sum exceeds the desired sum $c$, backtrack. I.e., do not enter its subtrees, go back to parent node.

• Keep a variable $r$ that gives you the sum of the numbers not yet considered. When you move to a right child, check if current subset sum $+ r \geq c$. If not, backtrack.
Backtracking

• Space required is $O(\text{tree height})$.
• With effective bounding functions, large instances can often be solved.
• For some problems (e.g., 0/1 knapsack), the answer (or a very good solution) may be found quickly but a lot of additional time is needed to complete the search of the tree.
• Run backtracking for as much time as is feasible and use best solution found up to that time.
Branch And Bound

• Search the tree using a breadth-first search (FIFO branch and bound).

• Search the tree as in a bfs, but replace the FIFO queue with a stack (LIFO branch and bound).

• Replace the FIFO queue with a priority queue (least-cost (or max priority) branch and bound). The priority of a node $p$ in the queue is based on an estimate of the likelihood that the answer node is in the subtree whose root is $p$. 
Branch And Bound

- Space required is $O(\text{number of leaves})$.
- For some problems, solutions are at different levels of the tree (e.g., 16 puzzle).

1 2 3 4
5 6 7 8
9 10 11 12
13 14 15
Branch And Bound

- FIFO branch and bound finds solution closest to root.
- Backtracking may never find a solution because tree depth is infinite (unless repeating configurations are eliminated).

• Least-cost branch and bound directs the search to parts of the space most likely to contain the answer. So it could perform better than backtracking.