Rank

Rank of an element is its position in ascending key order.

\[ [2,6,7,8,10,15,18,20,25,30,35,40] \]

\[ \text{rank}(2) = 0 \]

\[ \text{rank}(15) = 5 \]

\[ \text{rank}(20) = 7 \]
Selection Problem

• Given \( n \) unsorted elements, determine the \( k \)'th smallest element. That is, determine the element whose rank is \( k-1 \).

• Applications
  - Median score on a test.
    - \( k = \text{ceil}(n/2) \).
  - Median salary of Computer Scientists.
  - Identify people whose salary is in the bottom 10%. First find salary at the 10% rank.
Selection By Sorting

- Sort the $n$ elements.
- Pick up the element with desired rank.
- $O(n \log n)$ time.
Divide-And-Conquer Selection

- Small instance has $n \leq 1$. Selection is easy.
- When $n > 1$, select a pivot element from out of the $n$ elements.
- Partition the $n$ elements into 3 groups left, middle and right as is done in quick sort.
- The rank of the pivot is the location of the pivot following the partitioning.
- If $k-1 = \text{rank}(\text{pivot})$, pivot is the desired element.
- If $k-1 < \text{rank}(\text{pivot})$, determine the $k$’th smallest element in left.
- If $k-1 > \text{rank}(\text{pivot})$, determine the $(k-\text{rank}(\text{pivot})-1)$’th smallest element in right.
D&C Selection Example

Find $k$th element of:

```
a      3 2 8 0 11 10 1 2 9 7 1
```

Use 3 as the pivot and partition.

```
a      1 2 1 0 2 3 10 11 9 7 8
```

$\text{rank(pivot)} = 5$. So pivot is the 6’th smallest element.
D&C Selection Example

If $k = 6$ ($k-1 = \text{rank(pivot)}$), pivot is the element we seek.

If $k < 6$ ($k-1 < \text{rank(pivot)}$), find $k'$th smallest element in left partition.

If $k > 6$ ($k-1 > \text{rank(pivot)}$), find $(k-\text{rank(pivot)}-1)'$th smallest element in right partition.
Time Complexity

- Worst case arises when the partition to be searched always has all but the pivot.
  - $O(n^2)$
- Expected performance is $O(n)$.
- Worst case becomes $O(n)$ when the pivot is chosen carefully.
  - Partition into $n/9$ groups with 9 elements each (last group may have a few more)
  - Find the median element in each group.
  - pivot is the median of the group medians.
  - This median is found using select recursively.
Closest Pair Of Points

- Given $n$ points in 2D, find the pair that are closest.
Applications

- We plan to drill holes in a metal sheet.
- If the holes are too close, the sheet will tear during drilling.
- Verify that no two holes are closer than a threshold distance (e.g., holes are at least 1 inch apart).
Air Traffic Control

• 3D -- Locations of airplanes flying in the neighborhood of a busy airport are known.

• Want to be sure that no two planes get closer than a given threshold distance.
Simple Solution

• For each of the $n(n-1)/2$ pairs of points, determine the distance between the points in the pair.
• Determine the pair with the minimum distance.
• $O(n^2)$ time.
Divide-And-Conquer Solution

• When \( n \) is small, use simple solution.
• When \( n \) is large
  ▪ Divide the point set into two roughly equal parts \( A \) and \( B \).
  ▪ Determine the closest pair of points in \( A \).
  ▪ Determine the closest pair of points in \( B \).
  ▪ Determine the closest pair of points such that one point is in \( A \) and the other in \( B \).
  ▪ From the three closest pairs computed, select the one with least distance.
Example

- Divide so that points in A have $x$-coordinate $\leq$ that of points in B.
• Find closest pair in $A$.
• Let $d_1$ be the distance between the points in this pair.
Example

- Find closest pair in B.
- Let $d_2$ be the distance between the points in this pair.
Let $d = \min\{d_1, d_2\}$.

Is there a pair with one point in $A$, the other in $B$ and distance $< d$?
Example

- Candidates lie within $d$ of the dividing line.
- Call these regions $R_A$ and $R_B$, respectively.
Example

- Let $q$ be a point in $R_A$.
- $q$ need be paired only with those points in $R_B$ that are within $d$ of $q$.y.
• Points that are to be paired with $q$ are in a $d \times 2d$ rectangle of $R_B$ (comparing region of $q$).
• Points in this rectangle are at least $d$ apart.
So the comparing region of $q$ has at most 6 points.

So number of pairs to check is $\leq 6|R_A| = O(n)$. 
Time Complexity

- Create a sorted by \(x\)-coordinate list of points.
  - \(O(n \log n)\) time.
- Create a sorted by \(y\)-coordinate list of points.
  - \(O(n \log n)\) time.
- Using these two lists, the required pairs of points from \(R_A\) and \(R_B\) can be constructed in \(O(n)\) time.
- Let \(n < 4\) define a small instance.
Time Complexity

- Let $t(n)$ be the time to find the closest pair (excluding the time to create the two sorted lists).
- $t(n) = c$, $n < 4$, where $c$ is a constant.
- When $n \geq 4$,
  
  $$t(n) = t(\text{ceil}(n/2)) + t(\text{floor}(n/2)) + an,$$

  where $a$ is a constant.
- To solve the recurrence, assume $n$ is a power of 2 and use repeated substitution.
- $t(n) = O(n \log n)$. 